FUNCTIONS OF BAIRE CLASS ONE

DENNY H. LEUNG AND WEE-KEE TANG

Abstract. Let K be a compact metric space. A real-valued function on K is said to be of Baire class one (Baire-1) if it is the pointwise limit of a sequence of continuous functions. In this paper, we study two well known ordinal indices of Baire-1 functions, the oscillation index β and the convergence index γ. It is shown that these two indices are fully compatible in the following sense : a Baire-1 function f satisfies $\beta(f) \leq \omega^{\xi_1} \cdot \omega^{\xi_2}$ for some countable ordinals ξ_1 and ξ_2 if and only if there exists a sequence of Baire-1 functions (f_n) converging to f pointwise such that $\sup_n \beta(f_n) \leq \omega^{\xi_1}$ and $\gamma((f_n)) \leq \omega^{\xi_2}$. We also obtain an extension result for Baire-1 functions analogous to the Tietze Extension Theorem. Finally, it is shown that if $\beta(f) \leq \omega^{\xi_1}$ and $\beta(g) \leq \omega^{\xi_2}$, then $\beta(fg) \leq \omega^{\xi}$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. These results do not assume the boundedness of the functions involved.

1. Preliminaries

Let K be a compact metric space. A function $f : K \to \mathbb{R}$ is said to be of *Baire class one*, or simply, *Baire-1*, if there exists a sequence (f_n) of real-valued continuous functions that converges pointwise to f. Let $\mathfrak{B}_1(K)$ (respectively, $\mathcal{B}_1(K)$) be the set of all real-valued (respectively, bounded real-valued) Baire-1 functions on K. Several authors have studied Baire-1 functions in terms of ordinal ranks associated to each function. (See, e.g., [2], [3] and [4]). In this paper, we study the relationship between two of these ordinal ranks, namely the oscillation rank β and the convergence rank γ.

We begin by recalling the definitions of the indices β and γ. Suppose that H is a compact metric space, and f is a real-valued function whose domain contains H. For any $\varepsilon > 0$, let $H^0(f,\varepsilon) = H$. If $H^\alpha(f,\varepsilon)$ is defined for some countable ordinal α, let $H^{\alpha+1}(f,\varepsilon)$ be the set of all those $x \in H^\alpha(f,\varepsilon)$ such that for every open set U containing x, there are two points x_1 and x_2 in $U \cap H^\alpha(f,\varepsilon)$ with $|f(x_1) - f(x_2)| \geq \varepsilon$. For a countable limit ordinal α, we let

$$H^\alpha(f,\varepsilon) = \bigcap_{\alpha'<\alpha} H^{\alpha'}(f,\varepsilon).$$

The index $\beta_H(f,\varepsilon)$ is taken to be the least α with $H^\alpha(f,\varepsilon) = \emptyset$ if such α exists, and ω_1 otherwise. The **oscillation index** of f is

$$\beta_H(f) = \sup\{\beta_H(f,\varepsilon) : \varepsilon > 0\}.$$

If the ambient space H is clear from the context, we write $\beta(f,\varepsilon)$ and $\beta(f)$ in place of $\beta_H(f,\varepsilon)$ and $\beta_H(f)$ respectively.

The γ index is defined analogously. If (f_n) is a sequence of real-valued functions such that $H \subseteq \bigcap_n \operatorname{dom}(f_n)$, let $H^0((f_n),\varepsilon) = H$ for any $\varepsilon > 0$. If $H^\alpha((f_n),\varepsilon)$

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has been defined for some countable ordinal α, let $H^{\alpha+1}\left(\left(f_n\right),\varepsilon\right)$ be the set of all those $x \in H^{\alpha}\left(\left(f_n\right),\varepsilon\right)$ such that for every open set U containing x and any $m \in \mathbb{N}$, there are two integers $n_1,\ n_2$ with $n_1 > n_2 > m$ and $x' \in U \cap H^{\alpha}\left(\left(f_n\right),\varepsilon\right)$ such that $\left|f_{n_1}\left(x'\right) - f_{n_2}\left(x'\right)\right| \geq \varepsilon$. Define

$$H^{\alpha}\left(\left(f_n\right),\varepsilon\right) = \bigcap_{\alpha'<\alpha} H^{\alpha'}\left(\left(f_n\right),\varepsilon\right)$$

if α is a countable limit ordinal. Let $\gamma_H\left(\left(f_n\right),\varepsilon\right)$ be the least α with $H^{\alpha}\left(\left(f_n\right),\varepsilon\right) = \emptyset$ if such α exists, and ω_1 otherwise. Finally, the **convergence index** of (f_n) is the ordinal

$$\gamma_H\left(\left(f_n\right)\right) = \sup\left\{\gamma_H\left(\left(f_n\right),\varepsilon\right) : \varepsilon > 0\right\}.$$

Again, if there is no ambiguity about the space H, we write $\gamma((f_n),\varepsilon)$ and $\gamma((f_n))$ for $\gamma_H((f_n),\varepsilon)$ and $\gamma_H((f_n))$ respectively.

It is known that a function $f : K \to \mathbb{R}$ is Baire-1 if and only if $\beta\left(f\right) < \omega_1$. (See [3, Proposition 1.2].) Following [3], we define the set of functions of small Baire class ξ and the set of *bounded* functions of small Baire class ξ for each countable ordinal ξ as

$$\mathfrak{B}_1^{\xi}\left(K\right) = \left\{f \in \mathfrak{B}_1\left(K\right) : \beta\left(f\right) \leq \omega^{\xi}\right\}$$

and

$$\mathcal{B}_1^{\xi}\left(K\right) = \left\{f \in \mathcal{B}_1\left(K\right) : \beta\left(f\right) \leq \omega^{\xi}\right\}$$

respectively. In [4], the following results are shown.

Theorem 1.1. *Let K be a compact metric space.*

1. [4, Theorem 7] *If ξ is a finite ordinal, then a function $f \in \mathcal{B}_1^{\xi+1}\left(K\right)$ if and only if there exists a sequence (f_n) in $\mathcal{B}_1^1\left(K\right)$ converging pointwise to f such that $\gamma((f_n)) \leq \omega^{\xi}$.*
2. [4, Corollary 9] *If ξ is an infinite countable ordinal, and $f \in \mathcal{B}_1\left(K\right)$ is the pointwise limit of a sequence (f_n) in $\mathcal{B}_1^1\left(K\right)$ such that $\gamma((f_n)) \leq \omega^{\xi}$, then $\beta(f) \leq \omega^{\xi}$.*

One of our main results generalizes and unifies the two parts of Theorem 1.1.

Theorem 1.2. *Let K be a compact metric space and let $\xi_1,\ \xi_2$ be countable ordinals. A function $f \in \mathfrak{B}_1^{\xi_1+\xi_2}\left(K\right)$, respectively, $\mathcal{B}_1^{\xi_1+\xi_2}\left(K\right)$, if and only if there exists a sequence (f_n) in $\mathfrak{B}_1^{\xi_1}\left(K\right)$, respectively, $\mathcal{B}_1^{\xi_1}\left(K\right)$, converging pointwise to f such that $\gamma((f_n)) \leq \omega^{\xi_2}$.*

In the course of proving Theorem 1.2, we show that any Baire-1 function f on a closed subspace H of a compact metric space K can be extended to a Baire-1 function g on K such that $\beta_H(f) = \beta_K(g)$ (Theorem 3.6). When $\beta_H(f) = 1$, this is the familiar Tietze Extension Theorem. Proposition 2.1 and Theorem 2.3 in [3] yield that for a bounded Baire-1 function f, $\beta(f)$ is the smallest ordinal ξ such that there exists a sequence of continuous functions (f_n) converging pointwise to f and having $\gamma((f_n)) = \xi$. Theorem 5.5 below shows that the same result holds without the boundedness assumption on the function f. In the last section, we consider the product of Baire-1 functions. In contrast to the class $\mathcal{B}_1^{\xi}(K)$, the class $\mathfrak{B}_1^{\xi}\left(K\right)$ is not closed under multiplication. Theorem 6.5 shows that if $f \in \mathfrak{B}_1^{\xi_1}\left(K\right)$

and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi}(K)$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. It is also shown that this result is the best possible.

Our notation is standard. In the sequel, K will always denote a compact metric space. If H is a closed subset of K, the derived set H' is the set of all limit points of H. A transfinite sequence of derived sets is defined in the usual manner. Let $H^{(0)} = H$ and $H^{(\alpha+1)} = \left(H^{(\alpha)}\right)'$ for any ordinal α. If α is a limit ordinal, let

$$H^{(\alpha)} = \bigcap_{\alpha' < \alpha} H^{(\alpha')}.$$

Given real-valued functions f and g defined on a set S, we let

$$\|f - g\|_S = \sup\{|f(s) - g(s)| : s \in S\}.$$

When there is no cause for confusion, we write $\|f - g\|$ for $\|f - g\|_S$. Since we shall be dealing with unbounded functions in general, this functional can take the value ∞ and is not a "norm". However, it is compatible with the topology of uniform convergence on the set $\mathbb{R}^S$ of all real-valued functions on S in the sense that the sets

$$U(f, \varepsilon) = \{g : \|g - f\|_S < \varepsilon\}$$

form a basis for the said topology.

2. Oscillation and convergence of Baire-1 functions

We begin by proving a result that yields an upper bound of the oscillation index of a Baire-1 function f as the product of the convergence index of a sequence of functions (f_n) converging pointwise to f, and the supremum of the oscillation indices of f_n's.

Lemma 2.1. *Let U and L be sets such that $U \subseteq L \subseteq K$, where U is open in K and L is closed in K. Suppose f, f_n $(n \geq 1)$ are Baire-1 functions on K, $\alpha < \omega_1$, and $\varepsilon > 0$. Then*
(a) $L^\alpha(f, \varepsilon) \subseteq K^\alpha(f, \varepsilon) \cap L$,
(b) $L^\alpha((f_n), \varepsilon) \subseteq K^\alpha((f_n), \varepsilon) \cap L$,
(c) $K^\alpha(f, \varepsilon) \cap U \subseteq L^\alpha(f, \varepsilon)$,
(d) $K^\alpha((f_n), \varepsilon) \cap U \subseteq L^\alpha((f_n), \varepsilon)$.

Proof. We only prove (c). The proof is by induction on α. The statement is trivial if $\alpha = 0$ or a limit ordinal. Suppose the statement is true for all ordinals not greater than α. Let $x \in K^{\alpha+1}(f, \varepsilon) \cap U$. If N is a neighborhood of x in K, then $N \cap U$ is open in K. Thus there exist $x_1, x_2 \in (N \cap U) \cap K^\alpha(f, \varepsilon) = N \cap (U \cap K^\alpha(f, \varepsilon)) \subseteq N \cap L^\alpha(f, \varepsilon)$ such that $|f(x_1) - f(x_2)| \geq \varepsilon$. Hence $x \in L^{\alpha+1}(f, \varepsilon)$. □

Proposition 2.2. *Let (f_n) be a sequence in $\mathfrak{B}_1(K)$ and let $\varepsilon > 0$. Suppose that $\beta(f_n, \varepsilon) \leq \beta_0$ for all $n \in \mathbb{N}$, and $\gamma((f_n), \varepsilon) \leq \gamma_0$. If (f_n) converges pointwise to a function f, then $\beta(f, 3\varepsilon) \leq \beta_0 \cdot \gamma_0$.*

Proof. We first consider the case $\gamma_0 = 1$. Then $K^1((f_n), \varepsilon) = \emptyset$. For each $x \in K$, there exist an open neighborhood U_x of x and $p_x \in \mathbb{N}$ such that whenever $n > m > p_x$,

$$|f_n(x') - f_m(x')| < \varepsilon$$

for all $x' \in U_x$. By the compactness of K, there exist $x_1, x_2, ..., x_k$ such that

$$K \subseteq \bigcup_{i=1}^{k} U_{x_i}.$$

Let $p_0 = \max\{p_{x_1}, p_{x_2}, ..., p_{x_k}\}$. Then for all $n > m > p_0$ and $y \in K$, we have $y \in U_{x_i}$ for some i, $1 \leq i \leq k$. Since $n > m > p_{x_i}$,

$$|f_n(y) - f_m(y)| < \varepsilon.$$

Taking limit as $n \to \infty$, we have

$$||f - f_m|| \leq \varepsilon \quad \text{for all } m > p_0. \tag{2.1}$$

Using (2.1), it is easy to verify by induction that

$$K^{\alpha}(f, 3\varepsilon) \subseteq K^{\alpha}(f_{p_0+1}, \varepsilon)$$

for all $\alpha < \omega_1$. In particular,

$$K^{\beta_0}(f, 3\varepsilon) \subseteq K^{\beta_0}(f_{p_0+1}, \varepsilon) = \emptyset.$$

Hence $\beta(f, 3\varepsilon) \leq \beta_0 = \beta_0 \cdot \gamma_0$.

Suppose the assertion is true for some γ_0. Let (f_n) be a sequence in $\mathfrak{B}_1(K)$ that converges pointwise to a function f. Suppose there exists $\varepsilon > 0$ such that $\beta(f_n, \varepsilon) \leq \beta_0$ for all $n \in \mathbb{N}$ and $\gamma((f_n), \varepsilon) \leq \gamma_0 + 1$. We need to show $\beta(f, 3\varepsilon) \leq \beta_0 \cdot (\gamma_0 + 1)$. Since $\gamma((f_n), \varepsilon) \leq \gamma_0 + 1$, we have $K^{\gamma_0+1}((f_n), \varepsilon) = \emptyset$. For each $m \in \mathbb{N}$, let U_m denote the $\frac{1}{m}-$neighborhood of $K^{\gamma_0}((f_n), \varepsilon)$. Denote $K \backslash U_m$ by $\tilde{K}_m$. From Lemma 2.1(a) and 2.1(b), for each $n \in \mathbb{N}$, $\beta_{\tilde{K}_m}(f_n, \varepsilon) \leq \beta_0$ and $\gamma_{\tilde{K}_m}((f_n), \varepsilon) \leq \gamma_0$. By the inductive hypothesis, we see that

$$\beta_{\tilde{K}_m}(f, 3\varepsilon) \leq \beta_0 \cdot \gamma_0.$$

From this and applying Lemma 2.1(c) with $U = K \setminus \overline{U}_m$, $L = \tilde{K}_m$ for all $m \in \mathbb{N}$, we see that $K^{\beta_0 \cdot \gamma_0}(f, 3\varepsilon) \subseteq K^{\gamma_0}((f_n), \varepsilon)$. Let

$$\tilde{K} = K^{\beta_0 \cdot \gamma_0}(f, 3\varepsilon) \subseteq K^{\gamma_0}((f_n), \varepsilon).$$

Then

$$\beta_{\tilde{K}}(f_n, \varepsilon) \leq \beta_0 \text{ and } \gamma_{\tilde{K}}((f_n), \varepsilon) = 1.$$

Thus

$$\beta_{\tilde{K}}(f, 3\varepsilon) \leq \beta_0 \text{ by the case when } \gamma_0 = 1.$$

Therefore

$$K^{\beta_0 \cdot (\gamma_0+1)}(f, 3\varepsilon) = K^{\beta_0 \cdot \gamma_0 + \beta_0}(f, 3\varepsilon) = \tilde{K}^{\beta_0}(f, 3\varepsilon) = \emptyset.$$

Hence

$$\beta(f, 3\varepsilon) \leq \beta_0 \cdot (\gamma_0 + 1).$$

Suppose $\gamma_0 < \omega_1$ is a limit ordinal and the statement holds for all ordinals $\gamma < \gamma_0$. Let $(f_n) \subseteq \mathfrak{B}_1(K)$ be a sequence that converges pointwise to a function f and let $\varepsilon > 0$ be given. Suppose that $\beta(f_n, \varepsilon) \leq \beta_0$ for all $n \in \mathbb{N}$, and $\gamma((f_n), \varepsilon) \leq \gamma_0$. Then $\gamma((f_n), \varepsilon) < \gamma_0$ and $\beta(f, 3\varepsilon) \leq \beta_0 \cdot \gamma((f_n), \varepsilon) < \beta_0 \cdot \gamma_0$. ∎

Theorem 2.3. *Let (f_n) be a sequence $\mathfrak{B}_1(K)$ converging pointwise to a function f. Suppose* $\sup\{\beta(f_n) : n \in \mathbb{N}\} \leq \beta_0$ *and $\gamma((f_n)) \leq \gamma_0$. Then f is Baire-1 and $\beta(f) \leq \beta_0 \cdot \gamma_0$.*

For the next corollary, recall that $DBSC(K)$ is the space of all differences of semicontinuous functions on K. It is known that $\mathcal{B}_1^1(K)$ is the closure of $DBSC(K)$ in the topology of uniform convergence ([3, Theorem 3.1]).

Corollary 2.4 ([4, Corollary 9]). *Let $f \in \mathcal{B}_1(K)$ be the pointwise limit of a sequence $(f_n) \subseteq DBSC(K)$. If $\gamma((f_n)) \leq \omega^\xi$, $\omega \leq \xi < \omega_1$, then $\beta(f) \leq \omega^\xi$.*

3. Extension of Baire-1 functions

In this section, we establish several results regarding the extension of Baire-1 functions. They are analogs of the Tietze Extension Theorem for continuous functions. These results are applied in the next section in proving the converse of Theorem 2.3.

Lemma 3.1. *Suppose that F is a closed subspace of K and that f is a Baire-1 function on F. For any $\varepsilon > 0$, there exists a continuous function $g : K \setminus F^1(f,\varepsilon) \to \mathbb{R}$ such that*

$$\|g - f\|_{F \setminus F^1(f,\varepsilon)} \leq \varepsilon.$$

Proof. For any $x \in F \setminus F^1(f,\varepsilon)$, choose an open neighborhood U_x of x in K such that $U_x \cap F^1(f,\varepsilon) = \emptyset$ and $|f(x_1) - f(x_2)| < \varepsilon$ for all $x_1, x_2 \in U_x \cap F$. The collection $\mathcal{U} = \{U_x : x \in F \setminus F^1(f,\varepsilon)\} \cup \{K \setminus F\}$ is an open cover of $K \setminus F^1(f,\varepsilon)$. By [1], Theorems IX.5.3 and VIII.4.2, there exists a partition of unity $(\varphi_U)_{U \in \mathcal{U}}$ subordinated to $\mathcal{U}$. If $U = U_x \in \mathcal{U}$ for some $x \in F \setminus F^1(f,\varepsilon)$, let $a_U = f(x)$; if $U = K \setminus F$, let $a_U = 0$. Define $g : K \setminus F^1(f,\varepsilon) \to \mathbb{R}$ by $g = \sum_{U \in \mathcal{U}} a_U \varphi_U$. The sum is well-defined since $\{\operatorname{supp} \varphi_U : U \in \mathcal{U}\}$ is locally finite. Let $x \in F \setminus F^1(f,\varepsilon)$. Then $\mathcal{V} = \{U \in \mathcal{U} : \varphi_U(x) \neq 0\}$ is a finite set, $\varphi_U(x) > 0$ for all $U \in \mathcal{V}$ and $\sum_{U \in \mathcal{V}} \varphi_U(x) = 1$. If $U \in \mathcal{V}$, then $x \in U \cap F$; hence $U \neq K \setminus F$. Therefore, $U = U_y$ for some $y \in F \setminus F^1(f,\varepsilon)$. But then $x, y \in U_y \cap F$ implies that $|a_U - f(x)| = |f(y) - f(x)| < \varepsilon$. It follows that

$$\begin{aligned} |g(x) - f(x)| &= \left| \sum_{U \in \mathcal{U}} a_U \varphi_U(x) - f(x) \right| = \left| \sum_{U \in \mathcal{V}} a_U \varphi_U(x) - \sum_{U \in \mathcal{V}} f(x) \varphi_U(x) \right| \\ &\leq \sum_{U \in \mathcal{V}} |a_U - f(x)| \varphi_U(x) < \varepsilon. \end{aligned}$$

This shows that

$$\|g - f\|_{F \setminus F^1(f,\varepsilon)} \leq \varepsilon.$$

Finally, if x is a point in $K \setminus F^1(f,\varepsilon)$, there exists an open neighborhood V of x in K such that $V \cap F^1(f,\varepsilon) = \emptyset$ and $\mathcal{W} = \{U \in \mathcal{U} : \operatorname{supp} \varphi_U \cap V \neq \emptyset\}$ is finite. Now

$$g_{|V} = \sum_{U \in \mathcal{U}} a_U \varphi_{U\,|V} = \sum_{U \in \mathcal{W}} a_U \varphi_{U\,|V}.$$

Hence $g_{|V}$ is continuous on V, since it is a finite linear combination of continuous functions. In particular, g is continuous at x. As $x \in K \setminus F^1(f,\varepsilon)$ is arbitrary, g is continuous on $K \setminus F^1(f,\varepsilon)$. ∎

Theorem 3.2. *Suppose that F is a closed subspace of K and that f is a Baire-1 function on F. For any $1 \leq \beta_0 < \omega_1$, and any $\varepsilon > 0$, there exists $g : K \backslash F^{\beta_0}(f,\varepsilon) \to \mathbb{R}$ such that*

$$\|g - f\|_{F \backslash F^{\beta_0}(f,\varepsilon)} \leq \varepsilon$$

and

$$\beta_H(g) \leq \beta_0 \text{ for all compact subsets } H \text{ of } K \setminus F^{\beta_0}(f,\varepsilon).$$

Proof. Let $h : K \setminus F^1(f,\varepsilon) \to \mathbb{R}$ be the function obtained from Lemma 3.1. If $1 \leq \alpha < \beta_0$, let $\tilde{K} = \tilde{F} = F^{\alpha}(f,\varepsilon)$. Applying Lemma 3.1 with $\tilde{K}$, $\tilde{F}$, and the function f yields a continuous function $g_\alpha : F^{\alpha}(f,\varepsilon) \setminus F^{\alpha+1}(f,\varepsilon) \to \mathbb{R}$ such that

$$\|g_\alpha - f\|_{F^{\alpha}(f,\varepsilon) \backslash F^{\alpha+1}(f,\varepsilon)} \leq \varepsilon.$$

Let $g = h \cup \left(\bigcup_{\alpha < \beta_0} g_\alpha\right) : K \setminus F^{\beta_0}(f,\varepsilon) \to \mathbb{R}$. Then $\|g - f\|_{F \backslash F^{\beta_0}(f,\varepsilon)} \leq \varepsilon$.

Suppose that $\delta > 0$ and H is a compact subset of $K \setminus F^{\beta_0}(f,\varepsilon)$. If $x \notin F^1(f,\varepsilon)$, then there exists an open neighborhood U of x such that

$$\overline{U} \cap F^1(f,\varepsilon) = \emptyset.$$

Note that $g_{|\overline{U}} = h_{|\overline{U}}$. By Lemma 2.1(c),

$$H^1(g,\delta) \cap U \subseteq \left(H \cap \overline{U}\right)^1(g,\delta) = \left(H \cap \overline{U}\right)^1(h,\delta) = \emptyset$$

by the continuity of h. In particular, $x \notin H^1(g,\delta)$. It follows that

$$H^1(g,\delta) \subseteq H \cap F^1(f,\varepsilon).$$

Repeating the argument inductively yields that

$$H^{\beta_0}(g,\delta) \subseteq H \cap F^{\beta_0}(f,\varepsilon) = \emptyset.$$

Hence $\beta_H(g) \leq \beta_0$, as required. □

We obtain the following corollaries by taking $F = K$ and $\beta_0 = \beta_F(f)$ respectively.

Corollary 3.3. *Let f be a Baire-1 function on K such that $\beta(f,\varepsilon) \leq \beta_0$ for some $1 \leq \beta_0 < \omega_1$ and $\varepsilon > 0$. Then there exists $g : K \to \mathbb{R}$ such that*

$$\|g - f\| \leq \varepsilon \text{ and } \beta(g) \leq \beta_0.$$

Corollary 3.4. *Let F be a closed subspace of K. If f is a Baire-1 function on F, then for every $\varepsilon > 0$ there exists a Baire-1 function g on K such that*

$$\|g - f\|_F \leq \varepsilon \text{ and } \beta_K(g) \leq \beta_F(f).$$

Next we show that Corollary 3.4 can be improved to an exact extension theorem (i.e., the case $\varepsilon = 0$). In the statement of Lemma 3.5, the vacuous sum $\sum_{j=1}^{0} g_j$ is taken to be the zero function.

Lemma 3.5. *Let F be a closed subspace of K and let f be a Baire-1 function on F. Then there exists a sequence of Baire-1 functions (g_n) on K such that*

(a) *g_n is continuous on $K \setminus F^1\left(f - \sum_{j=1}^{n-1} g_j, \frac{1}{2^{n-1}}\right)$ for all $n \in \mathbb{N}$,*

(b) *$\left\|f - \sum_{j=1}^{n} g_j\right\|_{F \backslash F^1\left(f, \frac{1}{4^{n-1}}\right)} \leq \dfrac{1}{2^{n-1}}$, $n \in \mathbb{N}$,*

(c) $\|g_n\|_K \leq \dfrac{1}{2^{n-2}}$ *if* $n \geq 2$, *and*

(d) $F^1\left(f - \sum_{j=1}^n g_j, \delta\right) \subseteq F^1\left(f, \frac{\delta}{2^n}\right)$ *if* $0 < \delta \leq \frac{1}{2^{n-2}}$, $n \in \mathbb{N}$.

Proof. The functions (g_n) are constructed inductively. By Lemma 3.1, there exists a continuous function $g_1 : K \setminus F^1(f,1) \to \mathbb{R}$ such that $\|f - g_1\|_{F\setminus F^1(f,1)} \leq 1$. Extend g_1 to a function on K by defining g_1 to be 0 on $F^1(f,1)$. Then (a) and (b) hold. Condition (c) holds vacuously. Moreover, if $x \in F \setminus F^1\left(f, \frac{\delta}{2}\right)$, $0 < \delta \leq 2$, then there exists a neighborhood U_1 of x in F such that $|f(x_1) - f(x_2)| < \frac{\delta}{2}$ for all x_1, $x_2 \in U_1$. Note that since $x \in F \setminus F^1\left(f, \frac{\delta}{2}\right)$, g_1 is continuous at x. Hence there exists a neighborhood U_2 of x in F such that $|g_1(x_1) - g_1(x_2)| < \frac{\delta}{2}$ for all x_1, $x_2 \in U_2$. Let $U = U_1 \cap U_2$. Then U is a neighborhood of x in F. For all x_1, $x_2 \in U$,

$$|(f - g_1)(x_1) - (f - g_1)(x_2)| < \delta.$$

Hence $x \notin F(f - g_1, \delta)$. This proves (d).

Suppose that g_1, g_2, ..., g_n have been chosen. By Lemma 3.1, there exists a continuous function $h : K \setminus F^1\left(f - \sum_{j=1}^n g_j, \frac{1}{2^n}\right) \to \mathbb{R}$ such that

$$\left\| f - \sum_{j=1}^n g_j - h \right\|_{F\setminus F^1(f - \sum_{j=1}^n g_j, \frac{1}{2^n})} \leq \frac{1}{2^n}.$$

Define $\tilde{h}$ on $K \setminus F^1(f - \sum_{j=1}^n g_j, \frac{1}{2^n})$ by $\tilde{h} = \left(h \wedge \dfrac{1}{2^{n-1}}\right) \vee \dfrac{-1}{2^{n-1}}$. Then $\tilde{h}$ is continuous on $K \setminus F^1(f - \sum_{j=1}^n g_j, \frac{1}{2^n})$. By (d), $F^1(f - \sum_{j=1}^n g_j, \frac{1}{2^n}) \subseteq F^1\left(f, \dfrac{1}{4^n}\right)$. Hence $\tilde{h}$ is defined and continuous on $K \setminus F^1\left(f, \dfrac{1}{4^n}\right)$. Moreover, it follows from (b) that

$$\left\| f - \sum_{j=1}^n g_j \right\|_{F\setminus F^1\left(f, \frac{1}{4^n}\right)} \leq \frac{1}{2^{n-1}}. \tag{3.1}$$

From inequality (3.1) and the definition of $\tilde{h}$, we have

$$\left\| f - \sum_{j=1}^n g_j - \tilde{h} \right\|_{F\setminus F^1\left(f, \frac{1}{4^n}\right)} \leq \left\| f - \sum_{j=1}^n g_j - h \right\|_{F\setminus F^1\left(f, \frac{1}{4^n}\right)}.$$

Therefore, $\left\| f - \sum_{j=1}^n g_j - \tilde{h} \right\|_{F\setminus F^1\left(f, \frac{1}{4^n}\right)} \leq \dfrac{1}{2^n}$. Now define

$$g_{n+1} = \begin{cases} \tilde{h} & \text{on } K \setminus F^1(f - \sum_{j=1}^n g_j, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}.$$

Then g_{n+1} is continuous on $K \setminus F^1(f - \sum_{j=1}^n g_j, \frac{1}{2^n})$. This proves (a). Furthermore,

$$\left\| f - \sum_{j=1}^{n+1} g_j \right\|_{F\setminus F^1\left(f, \frac{1}{4^n}\right)} = \left\| f - \sum_{j=1}^n g_j - \tilde{h} \right\|_{F\setminus F^1\left(f, \frac{1}{4^n}\right)} \leq \frac{1}{2^n}.$$

This proves (b). Also,

$$\|g_{n+1}\|_K \leq \left\|\tilde{h}\right\|_{K\setminus F^1(f-\sum_{j=1}^n g_j,\frac{1}{2^n})} \leq \frac{1}{2^{n-1}}$$

by the definition of $\tilde{h}$. This proves (c). Finally, suppose $0<\delta\leq \dfrac{1}{2^{n-1}}$. Assume that $x\in F\setminus F^1\left(f,\dfrac{\delta}{2^{n+1}}\right)$. Then $x\notin F^1\left(f-\sum_{j=1}^n g_j,\dfrac{\delta}{2}\right)$. Thus there exists a neighborhood U_1 of x in F such that

$$\left|\left(f-\sum_{j=1}^n g_j\right)(x_1)-\left(f-\sum_{j=1}^n g_j\right)(x_2)\right|<\frac{\delta}{2}$$

whenever $x_1,\,x_2\in U_1$. Note that since $x\in F\setminus F^1\left(f-\sum_{j=1}^n g_j,\dfrac{\delta}{2}\right)$, g_{n+1} is continuous at x. Therefore, there exists a neighborhood U_2 of x in F such that $|g_{n+1}(x_1)-g_{n+1}(x_2)|<\dfrac{\delta}{2}$ for all $x_1,\,x_2\in U_2$. Let $U=U_1\cap U_2$. Then U is a neighborhood of x in F such that

$$\left|\left(f-\sum_{j=1}^{n+1} g_j\right)(x_1)-\left(f-\sum_{j=1}^{n+1} g_j\right)(x_2)\right|<\delta$$

whenever $x_1,\,x_2\in U$. Hence $x\notin F^1\left(f-\sum_{j=1}^{n+1} g_j,\delta\right)$. This proves (d). □

Theorem 3.6. *Let F be a closed subspace of K and let f be a Baire-1 function on F. Then there exists a Baire-1 function g on K such that*

$$g_{|F}=f \text{ and } \beta(g)=\beta_F(f).$$

Proof. Let (g_n) be the sequence given by Lemma 3.5. Define g on K by

$$g=\begin{cases}\sum_{j=1}^\infty g_j & \text{on } K\setminus F\\ f & \text{on } F\end{cases}.$$

Note that by (c) of Lemma 3.5, $\sum_{j=1}^\infty g_j$ converges uniformly on K. Hence g is well defined. Obviously, $g_{|F}=f$.

Claim. $K^1\left(g,\frac{1}{2^{n-3}}\right)\subseteq F^1\left(f,\frac{1}{4^n}\right)$ for all $n\in\mathbb{N}$.

Proof of Claim. Let $x\in K\setminus F^1\left(f,\frac{1}{4^n}\right)$. We consider two cases. Suppose $x\notin F$. By Lemma 3.5(a), g_j is continuous on $K\setminus F$ for all j. Since $\sum_{j=1}^n g_j$ converges uniformly to g on $K\setminus F$, and $K\setminus F$ is an open subset of K, g is continuous at x. Hence $x\notin K^1\left(g,\frac{1}{2^{n-3}}\right)$. Now suppose $x\in F$. Then $x\in F\setminus F^1\left(f,\frac{1}{4^n}\right)$. There is a neighborhood U_1 of x in K such that $|f(x)-f(x')|<\dfrac{1}{4^n}$ for all $x'\in U_1\cap F$. Also, for $1\leq k\leq n$,

$$\begin{aligned}F^1\left(f-\sum_{j=1}^k g_j,\frac{1}{2^k}\right)&\subseteq F^1\left(f,\frac{1}{4^k}\right) \text{ by Lemma 3.5(d),}\\ &\subseteq F^1\left(f,\frac{1}{4^n}\right).\end{aligned}$$

Since g_{k+1} is continuous on $K \setminus F^1(f - \sum_{j=1}^{k} g_j, \frac{1}{2^k})$, g_{k+1} is continuous on $K \setminus F^1\left(f, \frac{1}{4^n}\right)$ for all k, $1 \le k \le n$. Similarly, $F^1(f,1) \subseteq F^1\left(f, \frac{1}{4^n}\right)$ and g_1 is continuous on $K \setminus F^1(f,1)$ by Lemma 3.5(a); thus, g_1 is continuous on $K \setminus F^1\left(f, \frac{1}{4^n}\right)$. Hence there exists a neighborhood U_2 of x in K such that $U_2 \subseteq K \setminus F^1\left(f, \frac{1}{4^n}\right)$ and

$$\left|\sum_{j=1}^{n+1} g_j(x') - \sum_{j=1}^{n+1} g_j(x)\right| < \frac{1}{2^n} \text{ for all } x' \in U_2.$$

Let $U = U_1 \cap U_2$. Then U is a neighborhood of x in K. If $x' \in U \cap F$, then $x' \in U_1 \cap F$. Thus $|g(x') - g(x)| = |f(x') - f(x)| < \dfrac{1}{4^n} < \dfrac{1}{2^{n-2}}$. If $x' \in U \setminus F$, then

$$\begin{aligned}
|g(x') - g(x)| &= \left|\sum_{j=1}^{\infty} g_j(x') - f(x)\right| \\
&\le \left|\sum_{j=1}^{n+1} g_j(x') - \sum_{j=1}^{n+1} g_j(x)\right| + \left|\sum_{j=1}^{n+1} g_j(x) - f(x)\right| + \left|\sum_{j=n+2}^{\infty} g_j(x')\right| \\
&< \frac{1}{2^n} + \left|\sum_{j=1}^{n+1} g_j(x) - f(x)\right| + \sum_{j=n+2}^{\infty} \|g_j\| \text{ since } x' \in U_2, \\
&\le \frac{1}{2^n} + \left|\sum_{j=1}^{n+1} g_j(x) - f(x)\right| + \sum_{j=n+2}^{\infty} \frac{1}{2^{j-2}}, \text{ by Lemma 3.5(c)}, \\
&\le \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^{n-1}}, \text{ by Lemma 3.5(b), since } x \in F \setminus F^1\left(f, \frac{1}{4^n}\right), \\
&= \frac{1}{2^{n-2}}.
\end{aligned}$$

Thus $|g(x') - g(x)| < \dfrac{1}{2^{n-2}}$ if $x' \in U$. Hence $|g(x_1) - g(x_2)| < \dfrac{1}{2^{n-3}}$ whenever x_1, $x_2 \in U$. Therefore $x \notin K^1\left(g, \dfrac{1}{2^{n-3}}\right)$. This proves the claim.

It follows by induction that $K^\alpha\left(g, \dfrac{1}{2^{n-3}}\right) \subseteq F^\alpha\left(f, \dfrac{1}{4^n}\right)$ for $1 \le \alpha < \omega_1$. Indeed, the Claim yields the assertion for $\alpha = 1$. If the inclusion holds for some α, $1 \le \alpha < \omega_1$, let $\tilde{F} = F^\alpha\left(f, \dfrac{1}{4^n}\right)$. Then $K^{\alpha+1}\left(g, \dfrac{1}{2^{n-3}}\right) \subseteq \tilde{F}^1\left(g, \dfrac{1}{2^{n-3}}\right) = \tilde{F}^1\left(f, \dfrac{1}{2^{n-3}}\right) \subseteq \tilde{F}^1\left(f, \dfrac{1}{4^n}\right) = F^{\alpha+1}\left(f, \dfrac{1}{4^n}\right)$. Hence the inclusion holds for $\alpha+1$. If the inclusion holds for all $1 \le \alpha' < \alpha$, where $\alpha < \omega_1$ is a limit ordinal, then

$$K^\alpha\left(g, \frac{1}{2^{n-3}}\right) = \bigcap_{1 \le \alpha' < \alpha} K^{\alpha'}\left(g, \frac{1}{2^{n-3}}\right) \subseteq \bigcap_{1 \le \alpha' < \alpha} F^{\alpha'}\left(f, \frac{1}{4^n}\right) = F^\alpha\left(f, \frac{1}{4^n}\right).$$

This proves the inclusion for $1 \le \alpha < \omega_1$. In particular, if $\beta_F(f) = \beta_0$, then $K^{\beta_0}\left(g, \dfrac{1}{2^{n-3}}\right) \subseteq F^{\beta_0}\left(f, \dfrac{1}{4^n}\right) = \emptyset$. Thus $\beta_K\left(g, \dfrac{1}{2^{n-3}}\right) \le \beta_0$ for all $n \in \mathbb{N}$.

Hence $\beta_K(g) \leq \beta_0$. Of course, since $g_{|F} = f$, $\beta_K(g) \geq \beta_F(f) \geq \beta_0$. Therefore $\beta_K(g) = \beta_0 = \beta_F(f)$. □

Remark 3.7. If $\beta_F(f) = 1$, Theorem 3.6 is the familiar Tietze Extension Theorem. If $\beta_F(f)$ is transfinite, the conclusion of Theorem 3.6 can be obtained easily by defining the extension g to be 0 on $K \setminus F$. However, we do not see a simple proof for finite $\beta_F(f)$.

4. Decomposition of Baire-1 functions

In this section, we give a proof of Theorem 1.2. The extension results in §3 are employed in the course of the proof.

Theorem 4.1. *Let f be a Baire-1 function on K, $1 \leq \beta_0, \gamma_0 < \omega_1$ and $\varepsilon > 0$. Then there exist*

$$\tilde{f} : K \setminus K^{\beta_0 \cdot \gamma_0}(f, \varepsilon) \to \mathbb{R}$$

and

$$f_n : K \setminus K^{\beta_0 \cdot \gamma_0}(f, \varepsilon) \to \mathbb{R}$$

such that (f_n) converges to f pointwise, $\left\|\tilde{f} - f\right\|_{K \setminus K^{\beta_0 \cdot \gamma_0}(f,\varepsilon)} \leq \varepsilon$ and $\beta_H(f_n) \leq \beta_0$, $\gamma_H((f_n)) \leq \gamma_0$ for all compact subsets H of $K \setminus K^{\beta_0 \cdot \gamma_0}(f, \varepsilon)$.

Proof. For $\alpha \leq \gamma_0$, let $K_\alpha = K^{\beta_0 \cdot \alpha}(f, \varepsilon)$. If $n \in \mathbb{N}$, let U_n^α be the $\dfrac{1}{n}$−neighborhood of K_α in K. For $\alpha < \gamma_0$, it follows from Theorem 3.2 that there exists $g_\alpha : K_\alpha \setminus K_{\alpha+1} \to \mathbb{R}$ such that $\|g_\alpha - f\|_{K_\alpha \setminus K_{\alpha+1}} \leq \varepsilon$ and $\beta_H(g_\alpha) \leq \beta_0$ for all compact subsets H of $K_\alpha \setminus K_{\alpha+1}$. List the ordinals in $[0, \gamma_0)$ in a (possibly finite) sequence $(\alpha_n)_{n=1}^p$. Here $p \in \mathbb{N}$ or $p = \infty$. For each $n \in \mathbb{N}$, let $F_n = \bigcup_{j=1}^{n \wedge p} \left(K_{\alpha_j} \setminus U_n^{\alpha_j+1}\right)$. Then F_n is a closed subset of K. It is also easy to see that $K_\alpha \setminus U_n^{\alpha+1}$ and $K_{\alpha'} \setminus U_n^{\alpha'+1}$ are disjoint if $\alpha \neq \alpha'$. Thus $\left(K_{\alpha_j} \setminus U_n^{\alpha_j+1}\right)_{j=1}^{n \wedge p}$ is a partition of F_n into clopen (in F_n) subsets. Now define $\tilde{g}_n : F_n \to K$ to be $\bigcup_{j=1}^{n \wedge p} g_{\alpha_j | K_{\alpha_j} \setminus U_n^{\alpha_j+1}}$. Since $H = K_{\alpha_j} \setminus U_n^{\alpha_j+1}$ is a compact subset of $K_{\alpha_j} \setminus K_{\alpha_j+1}$, $\beta_H\left(g_{\alpha_j}\right) \leq \beta_0$. From the clopeness of the partition $\left(K_{\alpha_j} \setminus U_n^{\alpha_j+1}\right)_{j=1}^{n \wedge p}$, it follows readily that $\beta_{F_n}(\tilde{g}_n) \leq \beta_0$. By Theorem 3.6, there exists a function f_n' on K such that $f'_{n|F_n} = \tilde{g}_n$ and $\beta_K(f_n') \leq \beta_0$. Finally, define f_n to be $f'_{n|K \setminus K_{\gamma_0}}$ and $\tilde{f}$ to be $\bigcup_{\alpha < \gamma_0} g_{\alpha | K_\alpha \setminus K_{\alpha+1}}$. It follows from the choices of the g_α's that $\left\|f - \tilde{f}\right\|_{K \setminus K_{\gamma_0}} \leq \varepsilon$. Since $\bigcup_{n=1}^\infty F_n = K \setminus K_{\gamma_0}$, $\lim f_n = \tilde{f}$ pointwise on $K \setminus K_{\gamma_0}$. Suppose H is a compact subset of $K \setminus K_{\gamma_0}$. Then $\beta_H(f_n) \leq \beta_K(f_n') \leq \beta_0$. To complete the proof, we claim that for any $\delta > 0$ and any $\gamma \leq \gamma_0$, $H^\gamma((f_n), \delta) \subseteq K_\gamma$. The proof of this is by induction on γ. The case $\gamma = 0$ and the limit case is trivial. Now assume that the claim holds for some $\gamma < \gamma_0$. Let $x \in H^\gamma((f_n), \delta) \setminus K_{\gamma+1}$. Choose $j_1, j_2 \in \mathbb{N}$ such that $\alpha_{j_1} = \gamma$ and $d(x, K_{\gamma+1}) \geq \dfrac{1}{j_2}$, where d is the metric on K. Denote $H^\gamma((f_n), \delta)$ by L and the

$\dfrac{1}{2j_0}$-ball in K centered at x by U, where $j_0 = \max\{j_1, 2j_2\}$. Note that $L \subseteq K_\gamma$ by the inductive hypothesis: For all $n \geq j_0 = \max\{j_1, 2j_2\}$,

$$L \cap U \subseteq L \cap \overline{U} \subseteq K_{\alpha_{j_1}} \setminus U_n^{\alpha_{j_1}+1} \subseteq F_n.$$

This implies that $f_{n|L\cap\overline{U}} = \tilde{g}_{n|L\cap\overline{U}} = g_{\alpha_{j_1}|L\cap\overline{U}} = g_{\gamma|L\cap\overline{U}}$ for all $n \geq j_0$. Thus $\left(L \cap \overline{U}\right)^1((f_n), \delta) = \emptyset$. By Lemma 2.1(d),

$$L^1((f_n), \delta) \cap (L \cap U) = \emptyset.$$

In particular,

$$x \notin L^1((f_n), \delta) = H^{\gamma+1}((f_n), \delta).$$

Since $x \in H^\gamma((f_n), \delta) \setminus K_{\gamma+1}$ is arbitrary, this shows that $H^{\gamma+1}((f_n), \delta) \subseteq K_{\gamma+1}$. □

In particular, if $\beta_K(f) \leq \beta_0 \cdot \gamma_0$, we have the following.

Theorem 4.2. *Let f be a Baire-1 function on K, $1 \leq \beta_0, \gamma_0 < \omega_1$, and $\beta(f) \leq \beta_0 \cdot \gamma_0$. For any $\varepsilon > 0$, there exist $\tilde{f} : K \to \mathbb{R}$ and a sequence of functions $f_n : K \to \mathbb{R}$ such that (f_n) converges to $\tilde{f}$ pointwise, $\left\|\tilde{f} - f\right\| \leq \varepsilon$, $\beta(f_n) \leq \beta_0$ for all $n \in \mathbb{N}$, and $\gamma((f_n)) \leq \gamma_0$.*

A couple more preparatory steps will allow us to improve Theorem 4.2 to an exact result (i.e., $\varepsilon = 0$) when γ_0 is of the right form.

Theorem 4.3 ([3, Lemma 2.5])**.** *If (f_n) and (g_n) are two sequences of real-valued functions on K such that $\gamma((f_n)) \leq \omega^\xi$ and $\gamma((g_n)) \leq \omega^\xi$ for some $\xi < \omega_1$, then $\gamma((f_n + g_n)) \leq \omega^\xi$.*

Proposition 4.4. *For $1 \leq \xi < \omega_1$, $\mathfrak{B}_1^\xi(K) = \left\{f \in \mathbb{R}^K : \beta(f) \leq \omega^\xi\right\}$ is a vector subspace of $\mathbb{R}^K$ that is closed under the topology uniform convergence.*

We postpone the proof of Proposition 4.4 until the next section. We are now ready to prove the converse of Theorem 2.3 in certain cases.

Theorem 4.5. *If $f \in \mathfrak{B}_1(K)$ and $\beta(f) \leq \beta_0 \cdot \omega^{\gamma_0}$ for some $1 \leq \beta_0 < \omega_1$ and $\gamma_0 < \omega_1$, then there exists $(f_n) \subseteq \mathfrak{B}_1(K)$ such that (f_n) converges pointwise to f, $\beta(f_n) \leq \beta_0$ for all $n \in \mathbb{N}$ and $\gamma((f_n)) \leq \omega^{\gamma_0}$.*

Proof. First we assume β_0 is of the form ω^{α_0}, where $\alpha_0 < \omega_1$. By Theorem 4.2 there exist a sequence $\left(f_n^1\right) \subseteq \mathfrak{B}_1(K)$ and a function $f^1 \in \mathfrak{B}_1(K)$ such that, $\beta\left(f_n^1\right) \leq \omega^{\alpha_0}$ for all n, $\left(f_n^1\right)$ converges pointwise to f^1, $\left\|f^1 - f\right\| \leq \dfrac{1}{2}$, and $\gamma\left(\left(f_n^1\right)\right) \leq \omega^{\gamma_0}$. Then $\beta\left(f^1\right) \leq \omega^{\alpha_0} \cdot \omega^{\gamma_0} = \omega^{\alpha_0+\gamma_0}$ by Theorem 2.3. This implies that $\beta\left(f - f^1\right) \leq \omega^{\alpha_0+\gamma_0}$ by Proposition 4.4. Hence there exist $\left(f_n^2\right) \subseteq \mathfrak{B}_1(K)$ and f^2 such that $\beta\left(f_n^2\right) \leq \omega^{\alpha_0}$ for all $n \in \mathbb{N}$, $\left(f_n^2\right)$ converges pointwise to f^2, $\left\|f - f^1 - f^2\right\| \leq \dfrac{1}{2^2}$, and $\gamma\left(\left(f_n^2\right)\right) \leq \omega^{\gamma_0}$. We may assume that $\left\|f_n^2\right\| \leq \dfrac{1}{2}$ for all $n \in \mathbb{N}$, for otherwise, simply replace f_n^2 by $\hat{f}_n^2 = \left(f_n^2 \vee \frac{-1}{2}\right) \wedge \frac{1}{2}$. Continuing, we obtain f^m and $(f_n^m)_{n=1}^\infty$ for each m such that

- $\|f_n^m\| \leq \dfrac{1}{2^{m-1}}$,
- $\beta(f_n^m) \leq \omega^{\alpha_0}$ for all $m, n \in \mathbb{N}$,
- $\gamma((f_n^m)_n) \leq \omega^{\gamma_0}$ for all $m \in \mathbb{N}$,
- $f^m = \lim_n f_n^m$ (pointwise) for all $m \in \mathbb{N}$, and
- $\sum_{m=1}^{\infty} f^m$ converges uniformly to f on K.

Let $g_n^m = f_n^1 + f_n^2 + ... + f_n^m$ and $g_n = \sum_{m=1}^{\infty} f_n^m$. By Theorem 4.3, $\gamma((g_n^m)_n) \leq \omega^{\gamma_0}$ for all $m \in \mathbb{N}$. Given $\varepsilon > 0$, there exists m_0 such that for all $n \in \mathbb{N}$, $\|g_n^{m_0} - g_n\| \leq \varepsilon$. Then $K^{\omega^{\gamma_0}}((g_n), 3\varepsilon) \subseteq K^{\omega^{\gamma_0}}((g_n^{m_0}), \varepsilon) = \emptyset$. Therefore $\gamma((g_n)) \leq \omega^{\gamma_0}$. By Proposition 4.4, $\beta(g_n^m) \leq \omega^{\alpha_0}$ for all m, n. Therefore, $\beta(g_n) \leq \omega^{\alpha_0}$ by Proposition 4.4. Moreover,

$$\begin{aligned}\lim_n g_n &= \lim_n \lim_m g_n^m = \lim_m \lim_n g_n^m \\ &= \lim_m \sum_{k=1}^{m} f^k = f \text{ pointwise.}\end{aligned}$$

This proves the theorem in case $\beta_0 = \omega^{\alpha_0}$, with (g_n) in place of (f_n).

For a general nonzero countable ordinal β_0, write β_0 in Cantor normal form as

$$\beta_0 = \omega^{\beta_1} \cdot m_1 + \omega^{\beta_2} \cdot m_2 + ... + \omega^{\beta_k} \cdot m_k,$$

where $k, m_1, ..., m_k \in \mathbb{N}, \omega_1 > \beta_1 > \beta_2 > ... > \beta_k$. If $\gamma_0 \neq 0$, then $\beta_0 \cdot \omega^{\gamma_0} = \omega^{\beta_1} \cdot \omega^{\gamma_0}$. By the previous case, there exists $(f_n) \subseteq \mathfrak{B}_1(K)$ such that $\beta(f_n) \leq \omega^{\beta_1} \leq \beta_0$, $\gamma((f_n)) \leq \omega^{\gamma_0}$ and (f_n) converges pointwise to f. If $\gamma_0 = 0$, take $f_n = f$ for all n. Then $\beta(f_n) \leq \beta_0$ for all n, $\gamma((f_n)) = 1 = \omega^{\gamma_0}$ and (f_n) converges pointwise to f. □

The combination of Theorem 2.3 and Corollary 4.6 yields Theorem 1.2.

Corollary 4.6. *Let $f \in \mathfrak{B}_1^{\xi}(K)$, respectively, $\mathcal{B}_1^{\xi}(K)$, for some $\xi < \omega_1$. For all countable ordinals μ, ν such that $\mu + \nu \geq \xi$, there exists a sequence $(f_n) \subseteq \mathfrak{B}_1^{\mu}(K)$, respectively, $\mathcal{B}_1^{\mu}(K)$, such that $f_n \to f$ pointwise, and $\gamma((f_n)) \leq \omega^{\nu}$.*

We do not know if Theorem 4.5 holds without the restriction on the form of the ordinal $\gamma((f_n))$.

Problem 4.7. *Is it true that if $f \in \mathfrak{B}_1(K)$ with $\beta(f) \leq \beta_0 \cdot \gamma_0$ for some countable ordinals β_0 and γ_0, then there exists a sequence (f_n) converging pointwise to f so that $\sup_n \beta(f_n) \leq \beta_0$ and $\gamma((f_n)) \leq \gamma_0$?*

As another application of our results, we give the proof of another characterization of the classes $\mathcal{B}_1^{\xi}(K)$ due to Kechris and Louveau.

Definition 4.8 ([3, Section 3])**.** *A family $\{\Phi_\xi : 0 \leq \xi < \omega_1\}$ of real-valued functions on K is defined as follows.*

$$\Phi_0 = C(K),$$

$$\Phi_{\xi+1} = \left\{ \begin{array}{c} f : f \text{ is the pointwise limit of a bounded sequence} \\ (f_n) \subseteq \Phi_\xi \text{ such that } \gamma((f_n)) \leq \omega. \end{array} \right\},$$

and for limit ordinals λ,

$$\Phi_\lambda = \left\{ \begin{array}{c} f : f \text{ is the uniform limit of a bounded sequence} \\ (f_n) \subseteq \bigcup_{\xi<\lambda} \Phi_\xi. \end{array} \right\}.$$

Corollary 4.9 ([3, Theorem 4.2])**.** *For each $\xi < \omega_1$, $\mathcal{B}_1^{\xi}(K) = \Phi_{\xi}$.*

Proof. The case $\xi = 0$ is trivial. Suppose the corollary holds for some $\xi < \omega_1$. If $f \in \mathcal{B}_1^{\xi+1}(K)$, it follows from Corollary 4.6 that f is the pointwise limit of a bounded sequence (f_n) in $\mathcal{B}_1^{\xi}(K)$ such that $\gamma((f_n)) \leq \omega$. Since $\mathcal{B}_1^{\xi}(K) = \Phi_{\xi}$ by the inductive hypothesis, $f \in \Phi_{\xi+1}$. Conversely, if $f \in \Phi_{\xi+1}$, then f is the pointwise limit of a sequence (f_n) in Φ_{ξ} with $\gamma((f_n)) \leq \omega$. Since $\Phi_{\xi} = \mathcal{B}_1^{\xi}(K)$, $\beta(f) \leq \omega^{\xi+1}$ by Theorem 2.3. Thus $f \in \mathcal{B}_1^{\xi+1}(K)$.

Now assume that the corollary holds for all $\xi' < \xi$, where ξ is a countable limit ordinal. Let $f \in \Phi_{\xi}$. By the inductive hypothesis, $\Phi_{\xi'} = \mathcal{B}_1^{\xi'}(K) \subseteq \mathcal{B}_1^{\xi}(K)$ for $\xi' < \xi$. Hence f is the uniform limit of a sequence in $\mathcal{B}_1^{\xi}(K)$, and thus belongs to $\mathcal{B}_1^{\xi}(K)$. Conversely, assume that $f \in \mathcal{B}_1^{\xi}(K)$. For every $n \in \mathbb{N}$, there exists $\xi_n < \xi$ such that $\beta\left(f, \frac{1}{n}\right) \leq \omega^{\xi_n}$. By Corollary 3.3, the exists $f_n \in \mathcal{B}_1^{\xi_n}(K) = \Phi_{\xi_n}$ such that $\|f - f_n\| \leq \frac{1}{n}$. Thus $f \in \Phi_{\xi}$, as required. $\square$

Remark 4.10. If a family $\{\Psi_{\xi} : 0 \leq \xi < \omega_1\}$ is defined in a similar way as the family $\{\Phi_{\xi} : 0 \leq \xi < \omega_1\}$ except for the removal of the boundedness condition on the sequence (f_n), then $\Psi_{\xi} = B_1^{\xi}(K)$ for all $\xi < \omega_1$.

5. Optimal limit of continuous functions

In this section we prove the equivalence of the indices β and γ for functions in $\mathfrak{B}_1(K)$ in the same sense that was established for $\mathcal{B}_1(K)$ in Theorem 2.3 of [3]. Namely, it is shown that for all $f \in \mathfrak{B}_1(K)$, $\beta(f)$ is the smallest ordinal γ_0 for which there exists a sequence (f_n) in $C(K)$ converging pointwise to f and satisfying $\gamma((f_n)) \leq \gamma_0$. Let us note that this result is also the converse of Theorem 2.3 when $\beta_0 = 1$.

Definition 5.1. *Let $(f_n) \subseteq \mathbb{R}^K$ and $f \in \mathbb{R}^K$. We write*

(a) $(g_n) \prec (f_n)$ if (g_n) is a convex block combination of (f_n), i.e., there exists a sequence of non-negative real numbers (a_k) and a strictly increasing sequence (p_n) in $\mathbb{N}$ such that $\sum_{k=p_{n-1}+1}^{p_n} a_k = 1$ and $g_n = \sum_{k=p_{n-1}+1}^{p_n} a_k f_k$ for all n $(p_0 = 0)$.

(b) $(g_n) \overset{a}{\prec} (f_n)$ if there exists $m \in \mathbb{N}$ such that $(g_n)_{n=m}^{\infty} \prec (f_n)$, and

(c) $[f]_{-M}^{M} = (f \vee -M) \wedge M$, where $0 \leq M \in \mathbb{R}$.

The easy proof of the next lemma is left to the reader.

Lemma 5.2. *If $(g_n) \overset{a}{\prec} (f_n)$, then $\gamma((g_n), \varepsilon) \leq \gamma((f_n), \varepsilon)$ for all $\varepsilon > 0$.*

Lemma 5.3. *Let f be a Baire-1 function on K. Suppose $\mathcal{H}$ is a countable collection of compact subsets of K such that $\|f\|_H < \infty$ for all $H \in \mathcal{H}$ and $\bigcup_{H \in \mathcal{H}} H = K$. Then there exists $(f_n) \subseteq C(K)$ such that*

(i) $f_n \to f$ pointwise, and

(ii) $\left(f_{n|H}\right)$ is a bounded subset of $C(H)$ for all $H \in \mathcal{H}$.

Proof. Write $\mathcal{H}$ as a sequence $(H_m)_{m=1}^{\infty}$. Without loss of generality, assume that $H_m \subseteq H_{m+1}$ for all $m \in \mathbb{N}$. Since f is Baire-1, there exists $\left(f_n^0\right) \subseteq C(K)$ such that $\left(f_n^0\right)$ converges pointwise to f. Assume that $\left(f_n^{m-1}\right)_n \subseteq C(K)$ has been chosen so that $\lim_n f_n^{m-1} = f$ pointwise. If $m, n \in \mathbb{N}$, let U_n^m be the $\dfrac{1}{n}-$neighborhood of H_m

in K and let $M_m = \|f\|_{H_m}$. For all n, the function $\left[f_n^{m-1}\right]_{-M_m|H_m}^{M_m} \cup f^{m-1}_{n|K\setminus U_n^m}$ is continuous on $H_m \cup (K \setminus U_n^m)$. Let f_n^m be a continuous extension of the function onto K. Then $(f_n^m) \subseteq C(K)$. If $x \in H_m$, then $\lim_n f_n^m(x) = \lim_n \left[f_n^{m-1}(x)\right]_{-M_m}^{M_m} = [f(x)]_{-M_m}^{M_m} = f(x)$ since $\|f\|_{H_m} = M_m$. If $x \notin H_m$, then there exists n_0 such that $x \in K \setminus U_{n_0}^m$; thus $x \in K \setminus U_n^m$ for all $n \geq n_0$. Therefore $f_n^m(x) = f_n^{m-1}(x)$ for all $n \geq n_0$. Hence $\lim_n f_n^m(x) = f(x)$. Thus $\lim_n f_n^m = f$ pointwise. Now for each $n \in \mathbb{N}$, let $f_n = f_n^n$. Since $H_m \subseteq H_n$ for all $n \geq m$, on H_m we have

$$\begin{aligned} f_n = f_n^n &= \left[f_n^{n-1}\right]_{-M_n}^{M_n} \\ &= \left[\left[f_n^{n-2}\right]_{-M_{n-1}}^{M_{n-1}}\right]_{-M_n}^{M_n} = \ldots = \left[\ldots\left[\left[f_n^{m-1}\right]_{-M_m}^{M_m}\right]_{-M_{m+1}}^{M_{m+1}}\ldots\right]_{-M_n}^{M_n} \\ &= \left[f_n^{m-1}\right]_{-M_m}^{M_m} \text{ as } M_m \leq M_{m+1} \leq \ldots \leq M_n. \end{aligned}$$

Thus $f_n = \left[f_n^{m-1}\right]_{-M_m}^{M_m}$ on H_m for all $n \geq m$. In particular, on the set H_m,

$$\lim_n f_n = \left[\lim_n f_n^{m-1}\right]_{-M_m}^{M_m} = [f]_{-M_m}^{M_m} = f$$

since $\|f\|_{H_m} = M_m$. As $K = \bigcup H_m$, we see that $f_n \to f$ pointwise. Also, for each m, $\left(f_{n|H_m}\right)_{n=m}^{\infty}$ is bounded (by M_m) in $C(H_m)$; thus $\left(f_{n|H_m}\right)_{n=1}^{\infty}$ is bounded in $C(H_m)$. □

For the next lemma, recall that for a real-valued function f defined on a set S, $\operatorname{osc}(f, S) = \sup\{|f(s_1) - f(s_2)| : s_1, s_2 \in S\}$.

Lemma 5.4. *Let (f_n) be bounded in $C(H)$, where H is a compact metric space. Suppose (f_n) converges pointwise to f and $H^1(f, \varepsilon) = \emptyset$ for some $\varepsilon > 0$, then there exists $(g_n) \prec (f_n)$ such that $H^1((g_n), 7\varepsilon) = \emptyset$.*

Proof. By Corollary 3.3, there exists $\tilde{f} \in C(H)$ such that $\left\|f - \tilde{f}\right\|_H \leq \varepsilon$. Then $\left(f_n - \tilde{f}\right)$ is bounded in $C(H)$, $f_n - \tilde{f} \to f - \tilde{f}$ pointwise and $\operatorname{osc}\left(f - \tilde{f}, H\right) \leq 2\varepsilon$. By the first statement in the proof of Theorem 2.3 in [3], there exists $(h_n) \prec \left(f_n - \tilde{f}\right)$ such that $\left\|h_n - (f - \tilde{f})\right\|_H \leq 3\varepsilon$. Let $g_n = h_n + \tilde{f}$ for all $n \in \mathbb{N}$. Then $(g_n) \prec (f_n)$ and $\|g_n - f\|_H \leq 3\varepsilon$ for all $n \in \mathbb{N}$. It follows that $H^1((g_n), 7\varepsilon) = \emptyset$. □

Theorem 5.5. *Let f be a Baire-1 function on K. There exists a sequence $(f_n) \subseteq C(K)$ such that (f_n) converges pointwise to f and $\gamma((f_n)) = \beta(f)$.*

Proof. Let $\beta_0 = \beta(f)$. For each $\alpha < \beta_0$, and all $m, j \in \mathbb{N}$, let $U_{m,j}^{\alpha}$ be the $\frac{1}{j}$−neighborhood of $K^{\alpha}\left(f, \frac{1}{m}\right)$ in K. Define

$$\mathcal{H} = \left\{K^{\alpha}\left(f, \frac{1}{m}\right) \setminus U_{m,j}^{\alpha+1} : \alpha < \beta_0,\, m, j \in \mathbb{N}\right\}.$$

Then $\mathcal{H}$ is a countable collection of compact subsets of K such that $\bigcup_{H \in \mathcal{H}} H = K$. If $\alpha < \beta_0$ and $m, j \in \mathbb{N}$, by Lemma 3.1, there is a continuous function g on $H = K^{\alpha}\left(f, \frac{1}{m}\right) \setminus U_{m,j}^{\alpha+1}$ such that $\|g - f\|_H \leq \frac{1}{m}$. Hence $\|f\|_H < \infty$ for all $H \in \mathcal{H}$.

By Lemma 5.3, there exists $(g_n) \subseteq C(K)$ such that (g_n) converges pointwise to f and $\left(g_{n|H}\right)$ is bounded in $C(H)$ for all $H \in \mathcal{H}$.

List the elements of $\mathcal{H}$ in a sequence $(H_k)_{k=1}^{\infty}$. Take $\varepsilon_k = \dfrac{1}{m}$ if H_k is of the form $K^{\alpha}\left(f, \dfrac{1}{m}\right) \setminus U_{m,j}^{\alpha+1}$ for some α, m, j. Let $\left(g_n^0\right) = (g_n)$. Suppose $\left(g_n^{k-1}\right)_n \prec (g_n)_n$ has been chosen. Then $\left(g_n^{k-1}\right)_n$ converges to f pointwise, $\left(g_{n|H_k}^{k-1}\right)$ is a bounded sequence in $C(H_k)$, and $(H_k)^1(f, \varepsilon_k) = \emptyset$. By Lemma 5.4, there exists $\left(g_n^k\right)_n \prec \left(g_n^{k-1}\right)_n$ such that $(H_k)^1\left(\left(g_n^k\right)_n, 7\varepsilon_k\right) = \emptyset$. Let $f_n = g_n^n$ for all $n \in \mathbb{N}$. Then $(f_n) \prec (g_n)$. Therefore $(f_n) \subseteq C(K)$ and (f_n) converges pointwise to f. We claim that for all $m \in \mathbb{N}$ and for all $\alpha \leq \beta_0$, $K^{\alpha}\left((f_n), \dfrac{7}{m}\right) \subseteq K^{\alpha}\left(f, \dfrac{1}{m}\right)$. We prove the claim by induction on α. The claim is trivial if $\alpha = 0$ or α is a limit ordinal. Assume that $\alpha \leq \beta_0$ is a successor ordinal and that the claim holds for $\alpha - 1$. Let $x \in K^{\alpha}\left((f_n), \dfrac{7}{m}\right)$. Then $x \in K^{\alpha-1}\left((f_n), \dfrac{7}{m}\right) \subseteq K^{\alpha-1}\left(f, \dfrac{1}{m}\right)$. If $x \notin K^{\alpha}\left(f, \dfrac{1}{m}\right)$, there exists $j \in \mathbb{N}$ such that $d\left(x, K^{\alpha}\left(f, \dfrac{1}{m}\right)\right) > \dfrac{1}{j}$. Choose k such that $H_k = K^{\alpha-1}\left(f, \dfrac{1}{m}\right) \setminus U_{m,j}^{\alpha}$. Then $(f_n) \overset{a}{\prec} \left(g_n^k\right)_n$ and $\gamma_{H_k}\left(\left(g_n^k\right)_n, 7\varepsilon_k\right) \leq 1$ since $(H_k)^1\left(\left(g_n^k\right)_n, 7\varepsilon_k\right) = \emptyset$. By Lemma 5.2, $(H_k)^1((f_n), 7\varepsilon_k) = \emptyset$. Thus $(H_k)^1\left((f_n), \dfrac{7}{m}\right) = \emptyset$. But since $d\left(x, K^{\alpha}\left(f, \dfrac{1}{m}\right)\right) > \dfrac{1}{j}$, there exists an open set U in $\tilde{K} = K^{\alpha-1}\left(f, \dfrac{1}{m}\right)$ such that $x \in U \subseteq H_k \subseteq \tilde{K}$. By Lemma 2.1(d), $\left(\tilde{K}\right)^1\left((f_n), \dfrac{7}{m}\right) \cap U \subseteq (H_k)^1\left((f_n), \dfrac{7}{m}\right) = \emptyset$. Therefore $x \notin \left(\tilde{K}\right)^1\left((f_n), \dfrac{7}{m}\right) = K^{\alpha}\left((f_n), \dfrac{7}{m}\right)$, a contradiction. This proves the claim. From the claim, $K^{\beta_0}\left((f_n), \dfrac{7}{m}\right) \subseteq K^{\beta_0}\left(f, \dfrac{1}{m}\right) = \emptyset$ for all $m \in \mathbb{N}$. Therefore $\gamma((f_n)) \leq \beta_0$. Since $\gamma((f_n)) \geq \beta_0$ by [3, Proposition 2.1], (or Theorem 2.3), $\gamma((f_n)) = \beta_0 = \beta(f)$. $\square$

Remark 5.6. Unlike in Theorem 2.3 of [3], in general we cannot get a sequence $(g_n) \prec (f_n)$ such that $\gamma((g_n)) = \beta(f)$. Indeed, let $K = [0,1]$ and for each $n \in N$ let f_n be a continuous function that vanishes outside $\left[\dfrac{1}{n+1}, \dfrac{1}{n}\right]$ such that $\int_K f_n = 1$. Then (f_n) converges pointwise to $f = 0$. Suppose $(g_n) \prec (f_n)$, then $\int_K g_n = 1$ for all $n \in N$. Thus (g_n) does not converge uniformly to f, i.e., $\gamma((g_n)) > 1 = \beta(f)$.

Proof of Proposition 4.4. It is easy to see that for all $f \in \mathfrak{B}_1^{\xi}(K)$ and $a \in \mathbb{R}$, $af \in \mathfrak{B}_1^{\xi}(K)$. If $f, g \in \mathfrak{B}_1^{\xi}(K)$, then by Theorem 5.5 there exist two sequences of continuous functions (f_n) and (g_n) converging pointwise to f and g respectively such that $\gamma((f_n)) \leq \omega^{\xi}$ and $\gamma((g_n)) \leq \omega^{\xi}$. According to Theorem 4.3, $\gamma((f_n + g_n)) \leq \omega^{\xi}$. Hence by Theorem 2.3, $f + g \in \mathfrak{B}_1^{\xi}(K)$. Finally, given $f \in \overline{\mathfrak{B}_1^{\xi}(K)}$ and $\varepsilon > 0$, choose $g \in \mathfrak{B}_1^{\xi}(K)$ such that $\|f - g\| \leq \dfrac{\varepsilon}{3}$. Then $K^{\omega^{\xi}}(f, \varepsilon) \subseteq K^{\omega^{\xi}}\left(g, \dfrac{\varepsilon}{3}\right) = \emptyset$. Thus $f \in \mathfrak{B}_1^{\xi}(K)$. $\square$

6. Product of Baire-1 functions

In [3], it is observed that the classes $\mathcal{B}_1^{\xi}(K)$, $\xi < \omega_1$ are closed under multiplication. However, it is relative easy to see that this fails for the classes $\mathfrak{B}_1^{\xi}(K)$. In this section, we show that if $f \in \mathfrak{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi}(K)$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. It is also shown that the result is sharp. The proof of the next lemma is left to the reader.

Lemma 6.1. *If f is bounded and $\gamma((g_n)) \leq \xi$, then $\gamma((fg_n)) \leq \xi$.*

Lemma 6.2. *If $f \in \mathcal{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi_1+\xi_2}(K)$.*

Proof. By Theorem 5.5, there exists a sequence $(g_n) \subseteq C(K)$ converging to g pointwise such that $\gamma((g_n)) = \omega^{\xi_2}$. For each $n \in \mathbb{N}$, $g_n \in C(K) \subseteq \mathcal{B}_1^{\xi_1}(K)$ and $f \in \mathcal{B}_1^{\xi_1}(K)$. By [3] (see the remark on [3, p. 217]), $fg_n \in \mathcal{B}_1^{\xi_1}(K)$. Lemma 6.1 implies that $\gamma((fg_n)) \leq \omega^{\xi_2}$. Since (fg_n) converges to fg pointwise, it follows from Theorem 2.3 that $\beta(fg) \leq \omega^{\xi_1+\xi_2}$, i.e., $fg \in \mathfrak{B}_1^{\xi_1+\xi_2}(K)$. □

Now suppose $f \in \mathfrak{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$. By Lemma 3.1, for all $\alpha < \omega^{\xi_2}$, there is a continuous function $g_\alpha : K^\alpha(g,1) \setminus K^{\alpha+1}(g,1) \to \mathbb{R}$ such that

$$\|g_\alpha - g\|_{K^\alpha(g,1)\setminus K^{\alpha+1}(g,1)} \leq 1.$$

Let $h = \bigcup_{\alpha<\omega^{\xi_2}} g_\alpha$. It follows from the proof of Theorem 3.2 that $\beta(h) \leq \omega^{\xi_2}$. Given a closed set $H \subseteq K$, we write

$$\mathrm{d}_f(H) = \left\{ x \in H : \limsup_{\substack{y\to x\\ y\in H}} |f(y)| = \infty \right\}.$$

It is easy to see that $\mathrm{d}_f(H)$ is a closed subset of H such that $\mathrm{d}_f(H) \subseteq H^1(f,\varepsilon)$ for any $\varepsilon > 0$.

Lemma 6.3. *Suppose that $\alpha < \omega_1$, $\delta > 0$ and $s > 2$. If $x \in \left[K \setminus K^1(g,1)\right] \cap K^\alpha(fh,\delta)$, then $x \in K^\alpha\left(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\right)$.*

Proof. The proof is by induction on α. The result is clear if $\alpha = 0$ or a limit ordinal. Assume that the lemma holds for some $\alpha < \omega_1$. Suppose $\delta > 0$ and $s > 2$ are given. Let $x \in \left[K \setminus K^1(g,1)\right] \cap K^{\alpha+1}(fh,\delta)$. If $x \in \mathrm{d}_f\left(K^\alpha\left(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\right)\right)$, then $x \in K^{\alpha+1}\left(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\right)$ and we are done. Otherwise, assume that $x \notin \mathrm{d}_f\left(K^\alpha\left(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\right)\right)$. Then there exist a neighborhood U_1 of x in K and $M < \infty$ such that $|f(y)| \leq M$ for all $y \in U_1 \cap K^\alpha\left(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\right)$. Since $h = g_0$ on $K \setminus K^1(g,1)$, and g_0 is continuous on $K \setminus K^1(g,1)$, there exists a neighborhood U_2 of x in K such that $|h(x_1) - h(x_2)| \leq \frac{\delta}{2M}$ and $2(|h(x_1)|+1) < s(|h(x)|+1)$ for all $x_1, x_2 \in U_2$. Set $U = (U_1 \cap U_2) \setminus K^1(g,1)$. Then U is a neighborhood of x.

Claim. $K^{\alpha}(fh,\delta)\cap U\subseteq K^{\alpha}\left(f,\dfrac{\delta}{s(|h(x)|+1)}\wedge 1\right)$.

Note that if $y\in U$, then $y\in U_2$. Hence there exists $t>2$ such that $t(|h(y)|+1)\leq s(|h(x)|+1)$. Also, $y\in K^{\alpha}(fh,\delta)\cap U$ implies that $y\in \left[K\setminus K^1(g,1)\right]\cap K^{\alpha}(fh,\delta)$. Thus $y\in K^{\alpha}\left(f,\dfrac{\delta}{t(|h(y)|+1)}\wedge 1\right)$ by the inductive hypothesis. Since

$$\frac{\delta}{t(|h(y)|+1)}\geq \frac{\delta}{s(|h(x)|+1)}\wedge 1,$$

$y\in K^{\alpha}\left(f,\dfrac{\delta}{s(|h(x)|+1)}\wedge 1\right)$, as required.

Now if V is a neighborhood of x in K, there exist $x_1,x_2\in U\cap V\cap\ K^{\alpha}(fh,\delta)$ such that

$$\begin{aligned}\delta&\leq |f(x_1)h(x_1)-f(x_2)h(x_2)|\\&\leq |f(x_1)-f(x_2)|\,|h(x_1)|+|h(x_1)-h(x_2)|\,|f(x_2)|\\&\leq |f(x_1)-f(x_2)|\,|h(x_1)|+\frac{\delta}{2M}\cdot M,\end{aligned}$$

where, in the last inequality, $|f(x_2)|\leq M$ since $x_2\in U\cap K^{\alpha}\left(f,\dfrac{\delta}{s(|h(x)|+1)}\wedge 1\right)$ by the claim. Therefore,

$$|f(x_1)-f(x_2)|\geq \frac{\delta}{s(|h(x)|+1)}\wedge 1.$$

By the claim, $x_1,x_2\in V\cap K^{\alpha}\left(f,\dfrac{\delta}{s(|h(x)|+1)}\wedge 1\right)$. Since V is an arbitrary neighborhood of x, this shows that

$$x\in K^{\alpha+1}\left(f,\frac{\delta}{s(|h(x)|+1)}\wedge 1\right).$$

This completes the induction. □

It follows from Lemma 6.3 that

$$K^{\omega^{\xi_1}}(fh,\delta)\subseteq K^1(g,1).$$

Repeating the argument in Lemma 6.3 inductively yields

Lemma 6.4. $K^{\omega^{\xi_1}\cdot\alpha}(fh,\delta)\subseteq K^{\alpha}(g,1)$ *for all* $\alpha<\omega_1$, *and* $\delta>0$.

In particular, $K^{\omega^{\xi_1}\cdot\omega^{\xi_2}}(fh,\delta)=\emptyset$ for all $\delta>0$, i.e., $fh\in\mathfrak{B}_1^{\xi_1+\xi_2}(K)$.

Theorem 6.5. *If* $f\in\mathfrak{B}_1^{\xi_1}(K)$ *and* $g\in\mathfrak{B}_1^{\xi_2}(K)$, *then* $fg\in\mathfrak{B}_1^{\xi}(K)$, *where* $\xi=\max\{\xi_1+\xi_2,\ \xi_2+\xi_1\}$.

Proof. From the above, we obtain a function h in K such that $\|g-h\|\leq 1$, $\beta(h)\leq\omega^{\xi_2}$ and $fh\in\mathfrak{B}_1^{\xi_1+\xi_2}(K)$. Since $g,h\in\mathfrak{B}_1^{\xi_2}(K)$, it follows from Proposition 4.4 that $g-h\in\mathfrak{B}_1^{\xi_2}(K)$. As $g-h$ is bounded, we see that $g-h\in\mathcal{B}_1^{\xi_2}(K)$. By Lemma 6.2, $(g-h)f\in\mathfrak{B}_1^{\xi_2+\xi_1}(K)\subseteq\mathfrak{B}_1^{\xi}(K)$. Also, $fh\in\mathfrak{B}_1^{\xi_1+\xi_2}(K)\subseteq\mathfrak{B}_1^{\xi}(K)$. Applying Proposition 4.4 again gives $fg=f(g-h)+fh\in\mathfrak{B}_1^{\xi}(K)$. □

Our final result shows that Theorem 6.5 is sharp. We omit the easy proof of the next lemma.

Lemma 6.6. *Suppose that $h \in \mathfrak{B}_1(K)$, $\alpha < \omega_1$, and $\varepsilon > 0$. Let $V = K \setminus K^\alpha(h,\varepsilon)$. For any $\eta < \omega_1$,*

$$K^\eta(h,\varepsilon) \setminus K^\alpha(h,\varepsilon) \subseteq K^\eta(h\chi_V,\varepsilon).$$

Theorem 6.7. *Suppose that ξ_1, ξ_2 are countable ordinals, and let*

$$\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}.$$

If K is a compact metric space such that $K^{(\omega^\xi)} \neq \emptyset$, then

$$\sup\left\{\beta(fg) : f \in \mathfrak{B}_1^{\xi_1}(K),\, g \in \mathfrak{B}_1^{\xi_2}(K)\right\} = \omega^\xi.$$

Proof. We may of course assume that neither ξ_1 nor ξ_2 is 0, and that $\xi = \xi_1 + \xi_2$. The assumption on K yields a $\{0,1\}$-valued function h in $\mathfrak{B}_1(K)$ such that $K^{\omega^\xi}(h,1) \neq \emptyset$. Denote $K^\alpha(h,1)$ by K_α, $\alpha < \omega_1$. Choose a sequence of ordinals $(\rho_k)_{k=0}^\infty$ with $\rho_0 = 0$ that strictly increases to ω^{ξ_1}. Let λ be any ordinal that is less than ω^{ξ_2}. Fix a function $u : [0,\omega^\lambda) \to \mathbb{N}$ such that $\left\{\alpha \in [0,\omega^\lambda) : u(\alpha) \leq k\right\}$ is finite for all $k \in \mathbb{N}$. Define real-valued functions f and g on K as follows. If $t \in K_{\omega^{\xi_1}\cdot\lambda}$, let $f(t) = g(t) = 0$. If $t \in K_{\omega^{\xi_1}\cdot\alpha+\rho_{k-1}} \setminus K_{\omega^{\xi_1}\cdot\alpha+\rho_k}$ for some $\alpha < \omega^\lambda$ and $k \in \mathbb{N}$, let $f(t) = \dfrac{h(t)}{ku(\alpha)}$ and $g(t) = ku(\alpha)$. Notice that $fg = h\chi_V$, where $V = K \setminus K^{\omega^{\xi_1}\cdot\lambda}(h,1)$. It follows from Lemma 6.6 that $K^\eta(h,1) \setminus K^{\omega^{\xi_1}\cdot\lambda}(h,1) \subseteq K^\eta(fg,1)$ for all $\eta < \omega_1$. Since $K^{\omega^\xi}(h,1) \neq \emptyset$, and $h \in \mathfrak{B}_1(K)$, $K^\eta(h,1) \setminus K^{\omega^{\xi_1}\cdot\lambda}(h,1) \neq \emptyset$ for all $\eta < \omega^{\xi_1}\cdot\lambda$. Thus $K^\eta(fg,1) \neq \emptyset$ for all $\eta < \omega^{\xi_1}\cdot\lambda$. Hence $\beta(fg) \geq \omega^{\xi_1}\cdot\lambda$.

We now turn to the calculation of $\beta(g)$ and $\beta(f)$. First notice that the sets $K_{\omega^{\xi_1}\cdot\alpha+\rho_{k-1}} \setminus K_{\omega^{\xi_1}\cdot\alpha+\rho_k}$, $k \in \mathbb{N}$, form a partition of $K_{\omega^{\xi_1}\cdot\alpha} \setminus K_{\omega^{\xi_1}\cdot(\alpha+1)}$ into relatively open sets for any $\alpha < \omega^\lambda$, and that g is constant on each set $K_{\omega^{\xi_1}\cdot\alpha+\rho_{k-1}} \setminus K_{\omega^{\xi_1}\cdot\alpha+\rho_k}$. Hence the restriction of g to $K_{\omega^{\xi_1}\cdot\alpha} \setminus K_{\omega^{\xi_1}\cdot(\alpha+1)}$ is a continuous function for each $\alpha < \omega^\lambda$. It follows readily by induction that for any $\varepsilon > 0$, $K^\alpha(g,\varepsilon) \subseteq K_{\omega^{\xi_1}\cdot\alpha}$ for all $\alpha \leq \omega^\lambda$. But $g = 0$ on $K_{\omega^{\xi_1}\cdot\alpha}$. Thus $K^{\omega^\lambda+1}(g,\varepsilon) = \emptyset$. Therefore $\beta(g) \leq \omega^\lambda + 1 \leq \omega^{\xi_2}$.

Finally, consider the function f. Let $k_0 \in \mathbb{N}$ be given. The set

$$A = \left\{(\alpha,k) : k \in \mathbb{N},\ \alpha \in [0,\omega^\lambda),\ ku(\alpha) \leq k_0\right\}$$

is finite. List the elements of A in a finite sequence $((\alpha_i,k_i))_{i=1}^j$ in lexicographical order. Then $|f(t_1) - f(t_2)| < \dfrac{1}{k_0}$ for all $t_1, t_2 \in K \setminus K_{\omega^{\xi_1}\cdot\alpha_1+\rho_{k_1-1}}$. Hence $K^1\left(f, \frac{1}{k_0}\right) \subseteq K_{\omega^{\xi_1}\cdot\alpha_1+\rho_{k_1-1}}$. Note that $f = \frac{h}{k_1 u(\alpha_1)}$ on $K_{\omega^{\xi_1}\cdot\alpha_1+\rho_{k_1-1}} \setminus K_{\omega^{\xi_1}\cdot\alpha_1+\rho_{k_1}}$. Thus $K^{1+\eta}\left(f, \frac{1}{k_0}\right) \subseteq K_{\omega^{\xi_1}\cdot\alpha_1+\rho_{k_1-1}+\eta}$ for all η such that $\omega^{\xi_1}\cdot\alpha_1 + \rho_{k_1-1} + \eta \leq \omega^{\xi_1}\cdot\alpha_1 + \rho_{k_1}$. Let η_0 be such that $\omega^{\xi_1}\cdot\alpha_1 + \rho_{k_1-1} + \eta_0 = \omega^{\xi_1}\cdot\alpha_1 + \rho_{k_1}$. Then $\eta_0 \leq \rho_{k_1}$. Therefore,

$$K^{1+\rho_{k_1}}\left(f, \tfrac{1}{k_0}\right) \subseteq K^{1+\eta_0}\left(f, \tfrac{1}{k_0}\right) \subseteq K_{\omega^{\xi_1}\cdot\alpha_1+\rho_{k_1}}.$$

Repeating the argument, we see that

$$K^\rho\left(f, \frac{1}{k_j}\right) \subseteq K_{\omega^{\xi_1}\cdot\alpha+\rho_{k_j}},$$

where $\rho = 1+\rho_{k_1}+1+\rho_{k_2}+...+1+\rho_{k_j}$. Since $0 \leq f(t) < \dfrac{1}{k_0}$ for all $t \in K_{\omega^{\xi_1}\cdot\alpha+\rho_{k_j}}$,

$$K^{\rho+1}\left(f, \frac{1}{k_j}\right) = \emptyset.$$

As (ρ_k) increases to ω^{ξ_1}, $\rho + 1 < \omega^{\xi_1}$. Hence $K^{\omega^{\xi_1}}\left(f, \dfrac{1}{k_0}\right) = \emptyset$ for any $k_0 \in \mathbb{N}$. It follows that $\beta(f) \leq \omega^{\xi_1}$. Summarizing, we have functions f and g such that $f \in \mathfrak{B}_1^{\xi_1}(K)$, $g \in \mathfrak{B}_1^{\xi_2}(K)$ and $\beta(fg) \geq \omega^{\xi_1} \cdot \lambda$. Since $\lambda < \omega^{\xi_2}$ is arbitrary, the theorem is proved. $\square$

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Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543.

E-mail address: `matlhh@nus.edu.sg`

Division of Mathematics, National Institute of Education, Nanyang Technological University, 469 Bukit Timah Road, Singapore 259756.

E-mail address: `wktang@nie.edu.sg`